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Saddlepoint Optimality in Differential Games

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A family of two-person zero-sum differential games in which the admissible strategies are Borel measurable is defined, and two types of saddlepoint conditions are introduced as optimality criteria. In one, saddlepoint candidates are compared at each point of the state space with all playable pairs at that point; and, in the other, they are compared only with strategy pairs playable on the entire state space. As a theorem, these two types of optimality are shown to be equivalent for the defined family of games. Also, it is shown that a certain closure property is sufficient for this equivalence. A game having admissible

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strategies everywhere constant in which the two types of saddlepoint candidates are not equivalent is discussed.

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SADDLEPOINT OPTIMALITY IN DIFFERENTIAL GAMES

INTRODUCTION

Let the strategy of the minimizing player be p and that of the maximizing player be e. Consider the question, What class of strategies e is to oppose a strategy, say p^*, when p^* is asserted to result in "optimum" performance for the minimizing player? As Isaacs [1] illustrates, this class must include all representable actions of the maximizing player. Of course, it suffices to include all strategies e that are "playable" with p^* at arbitrary points of the state (playing) space. But, is it enough to include only those strategies e that are playable with p^* on the entire state space? In this report we address this question and show under what conditions the answer is affirmative. In addition, a simple game is discussed which illustrates that the affirmative is not universal for all games.

The question does not arise in differential games where it is assumed a priori that all strategy pairs are "playable;" [2,3]. In general, this assumption is invalid for differential games where there are state constraints and/or a target set that is a function of the state variables other than time [4,5]. Even in games of prescribed duration or in pursuit—evasion games where capture time is optimized, one cannot guarantee the "playability" of all strategy pairs unless certain assumptions are made on the dynamics of the game; e.g., see Varaiya [6]. Consequently, our question is posed for the large class of games where not all strategy pairs are necessarily playable. We say that a strategy pair (p^*, e^*) is optimal of Type I iff it is a saddlepoint with respect to all pairs (p^*, e) and (p, e^*) that are playable on at least one point of the state space. This is a pointwise type of optimality. We say that a strategy pair (p^*, e^*) is optimal of Type II iff it is a saddlepoint with respect to all pairs (p^*, e) and (p, e^*) that are playable over the entire state space. This is a global type of optimality. With this terminology, our question is rephrased as, Is optimality of Type II equivalent to that of Type I?

In the next section a family of games is defined in which the admissible strategies are Borel measurable. The pointwise and global definitions of saddlepoint optimality are then given, and a theorem asserting the equivalence between these two types for the given family of games is proved. A closure property similar to that introduced in another paper [7] is described. This is a property on the class of admissible strategies; i.e., a function formed in a certain way from any two members of the admissible class is also admissible. It is pointed out that this closure property is a sufficient condition for equivalence in general games. A game example in which neither the closure property nor the equivalence holds is analyzed in the last section.

DEFINITION OF TWO-PERSON ZERO-SUM DIFFERENTIAL GAMES

Consider a differential game with state equations

$$\dot{x} = f(x, u, v) \qquad x \in E^n, \quad u \in E^r, \quad v \in E^s \tag{1}$$

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where f is a Borel-measurable function mapping $E^n \times E^r \times E^s$ into E^n. The playing space X is a Lebesgue-measurable subset of E^n. The target θ is a closed set contained in the closure of X.

The two players, one the minimizer and the other the maximizer, choose the values of u and v, respectively. Let U and V be compact subsets of E^r and E^s, respectively. Define the two spaces of functions P and E by

$$P = \{p{:}E^n \to U \mid \quad p \text{ is Borel measurable}\} \tag{2}$$

$$E = \{e{:}E^n \to V \mid \quad e \text{ is Borel measurable}\}\ . \tag{3}$$

The spaces P and E constitute the most general class of strategies considered herein from which admissible strategies are defined. Because the strategies of the players have X as their domain rather than E^n, we define the sets of admissible strategies $\hat{\mathcal{P}}$ and $\hat{\mathcal{E}}$ as

$$\hat{\mathcal{P}} = \{p{:}X \to U \mid \ \exists\ p_1 \in P \ \ni p_1|X = p\} \tag{4}$$

$$\hat{\mathcal{E}} = \{e{:}X \to V \mid \ \exists\ e_1 \in E \ \ni e_1|X = e\} \tag{5}$$

where the notation $p_1|X = p$ means that $p_1(x) = p(x)$ for all $x \in X$; i.e., p is the restriction of p_1 to X. An analogous meaning attaches to $e_1|X = e$.

Let $x_0 \in X$. A pair (p, e) with $p \in \hat{\mathcal{P}}$ and $e \in \hat{\mathcal{E}}$ is a *playable strategy pair* at x_0 if it generates at least one terminating trajectory ϕ satisfying

$$\phi(t) = x_0 + \int_{t_0}^{t} f(\phi(\tau), p(\phi(\tau)), e(\phi(\tau)))\, d\tau \qquad \forall t \in [t_0, t_f] \tag{6}$$

where $\phi(t) \in X$ for all $t \in [t_0, t_f)$ and where t_f is the first time for which $\phi(t_f) \in \theta$. Let $\mathcal{T}(x_0)$ denote the set of all playable strategy pairs at the point x_0. Define

$$\mathcal{T}(X) = \bigcap_{x_0 \in X} \mathcal{T}(x_0). \tag{7}$$

The set $\mathcal{T}(X)$ is the set of all strategy pairs that are playable at every initial point of the playing space. We assume that $\mathcal{T}(X)$ is nonempty.

Let $x_0 \in X$. Define

$$\mathcal{P}(x_0) = \{p \in \hat{\mathcal{P}} \mid \quad \exists\, e \in \hat{\mathcal{E}} \ni (p, e) \text{ is playable at } x_0\}$$

$$\mathcal{E}(x_0) = \{e \in \hat{\mathcal{E}} \mid \quad \exists\, p \in \hat{\mathcal{P}} \ni (p, e) \text{ is playable at } x_0\}.$$

For each $e \in \mathcal{E}(x_0)$, we define

$$\mathcal{P}(e, x_0) = \{p \in \mathcal{P}(x_0) \mid \quad (p, e) \text{ is playable at } x_0\}; \tag{8}$$

and for each $p \in \mathcal{P}(x_0)$, we define

$$\mathcal{E}(p, x_0) = \{e \in \mathcal{E}(x_0) | \quad (p, e) \text{ is playable at } x_0 \}. \tag{9}$$

The set $\mathcal{P}(x_0)$ is the largest set of strategies from which the minimizer is permitted to choose when the initial state is x_0. If the maximizer plays $e \in \mathcal{E}(x_0)$ for the initial state x_0, then the minimizer can only choose from the smaller space $\mathcal{P}(e, x_0)$. Observe that

$$\mathcal{P}(x_0) = \bigcup_{e \in \mathcal{E}(x_0)} \mathcal{P}(e, x_0)\,.$$

An analogous statement holds for $\mathcal{E}(x_0)$. Furthermore, we make the following definitions for each $e \in \mathcal{E}(x_0)$ and each $p \in \mathcal{P}(x_0)$, respectively:

$$\mathcal{P}(e, X) = \bigcap_{x_0 \in X} \mathcal{P}(e, x_0) \tag{10}$$

$$\mathcal{E}(p, X) = \bigcap_{x_0 \in X} \mathcal{E}(p, x_0)\,. \tag{11}$$

If the set $\mathcal{P}(e, X)$ is to be nonempty, it follows necessarily that e must have a playable mate at each $x_0 \in X$ and that it must have at least the same such mate for all $x_0 \in X$. An analogous statement holds for $\mathcal{E}(p, X)$.

Let $x_0 \in X$ and $(p, e) \in \mathcal{T}(x_0)$. We define $T(x_0; p, e)$ to be the set of all solutions ϕ of Eq. (1) emanating from x_0 and due to the strategy pair (p, e). Let $\phi \in T(x_0; p, e)$. A real number denoted by $V(x_0; p, e, \phi)$ is associated with the quadruple $\{x_0, p, e, \phi\}$ by

$$V(x_0; p, e, \phi) = \int_{t_0}^{t_f} f_0(\phi(\tau), p(\phi(\tau)), e(\phi(\tau)))\, d\tau \tag{12}$$

where f_0 is a real-value, bounded, Borel-measurable function with domain $E^n \times E^r \times E^s$. For each $x_0 \in X$, the minimizer desires to minimize the number defined by Eq. (12), while the maximizer wants to maximize it.

The concept of optimality for this game is explored in the next section.

POINTWISE AND GLOBAL DEFINITIONS OF SADDLEPOINT OPTIMALITY

In this section we define two types of saddlepoint conditions of optimality. The first one, Type I, is termed *pointwise* because a candidate strategy pair for optimality is compared at each point of the playing space with all strategy pairs that are playable at that point. In Type II, such a candidate pair is compared only with those strategy pairs that are playable over the entire state space, hence the term *global*. While these two types are equivalent (Theorem 1 in the next section) for Borel strategies, they are not necessarily so for other classes of strategies (as established by the example in the Discussion section). These two types of saddlepoint optimality are defined as follows.

Saddlepoint Optimality I. A strategy pair (p^*, e^*) is optimal iff it belongs to $\mathcal{T}(X)$ and, for all $x_0 \in X$ and for all $\phi^* \in T(x_0; p^*, e^*)$, the following two inequalities are met:

$$V(x_0; p^*, e^*, \phi^*) \leqslant V(x_0; p, e^*, \phi) \qquad \forall p \in \mathcal{P}(e^*, x_0), \qquad \forall \phi \in T(x_0; p, e^*) \quad (13)$$

$$V(x_0; p^*, e, \phi) \leqslant V(x_0; p^*, e^*, \phi^*) \qquad \forall e \in \mathcal{E}(p^*, x_0), \qquad \forall \phi \in T(x_0; p^*, e)\,. \quad (14)$$

Saddlepoint Optimality II. A strategy pair (p^*, e^*) is optimal iff it belongs to $\mathcal{T}(X)$ and, for all $x_0 \in X$ and for all $\phi^* \in T(x_0; p^*, e^*)$, the following two inequalities are met:

$$V(x_0; p^*, e^*, \phi^*) \leqslant V(x_0; p, e^*, \phi) \qquad \forall p \in \mathcal{P}(e^*, X), \qquad \forall \phi \in T(x_0; p, e^*) \quad (15)$$

$$V(x_0; p^*, e, \phi) \leqslant V(x_0; p^*, e^*, \phi^*) \qquad \forall e \in \mathcal{E}(p^*, X), \qquad \forall \phi \in T(x_0; p^*, e)\,. \quad (16)$$

The strategies p and e as arguments in Eqs. (15) and (16) do not depend on the initial state for their playability; those in Eqs. (13) and (14) do. Thus the saddlepoint optimality of Type II compares a candidate strategy pair (p^*, e^*) for optimality with strategy pairs from a considerably smaller class than that of Type I. Hence it is of interest to show under what conditions the two types are equivalent with respect to proclaiming the same candidates to be optimal. By way of Lemma 1 of the next section we show that these two types are equivalent for the game as formulated in the preceding section. Recall that the Borel strategies form the admissible strategies of the players. This Borel class of strategies satisfies a certain closure property which is discussed in the final section. This property of a set being "closed" provides a sufficient condition for the equivalence of the two types of saddlepoint optimality. An example is discussed in which the closure property fails to hold and where, indeed, a candidate is optimal of Type II but not of Type I.

A THEOREM OF EQUIVALENCE

The following lemma is the basis for establishing the equivalence between the two types of saddlepoint optimality.

Lemma 1. *Let* $x_0 \in X$. *If* (p^*, e^*) *and* (p, e^*) *are playable at* x_0 *and if* $\phi \in T(x_0; p, e^*)$, *then the strategy pair* (p_1, e^*) *is playable at* x_0 *where* $p_1 : X \to U$ *is defined by*

$$p_1(x) = \begin{cases} p(x) & \forall x \in \{\phi(t) : t \in [t_0, t_f)\}, \text{ where } [t_0, t_f] \text{ is the interval of definition of } \phi \\ p^*(x) & \text{for all other } \; x \in X. \end{cases} \quad (17)$$

Furthermore, (p_1, e^*) is playable over the entire playing space provided (p^*, e^*) is.

Proof. Let $A = \{\phi(t):t \in [t_0, t_f]\}$. The set A is a compact subset of E^n because ϕ is absolutely continuous and its domain is compact. The continuous image of a compact set by a continuous function is compact [8]; thus, A is a Borel-measurable set.

According to the definitions of $\hat{\mathcal{P}}$ and $\hat{\mathcal{E}}$, there exist $\hat{p}^*$, $\hat{p} \in P$, and $\hat{e}^* \in E$ such that

$$p^*(x) = \hat{p}^*(x), \tag{18}$$

$$p(x) = \hat{p}(x), \tag{19}$$

$$e^*(x) = \hat{e}^*(x) \tag{20}$$

for all $x \in X$. Define $\hat{p}_1:E^n \to U$ as

$$\hat{p}_1(x) = \begin{cases} \hat{p}(x) & \forall x \in A \\ \hat{p}^*(x) & \forall x \in E^n - A. \end{cases} \tag{21}$$

The function $\hat{p}_1$ is Borel measurable because $\hat{p}^*$ and $\hat{p}$ are Borel measurable and A is a Borel-measurable set. Consequently, $p_1 \in \hat{\mathcal{P}}$ because p_1 is the restriction of $\hat{p}_1$ to X. Because $p_1(x) = p(x)$ at all points x along the trajectory ϕ, it follows that ϕ is a terminating trajectory for the pair (p_1, e^*). Therefore, (p_1, e^*) is a playable strategy pair at x_0.

Suppose that (p^*, e^*) belongs to $\mathcal{T}(X)$. We want to show that (p_1, e^*) belongs to $\mathcal{T}(X)$. We have already shown that (p_1, e^*) belongs to $\mathcal{T}(x_0)$. Suppose to the contrary that $(p_1, e^*) \notin \mathcal{T}(X)$. Then there is some $x \in X$ with $x \neq x_0$ such that $(p_1, e^*) \notin \mathcal{T}(x)$; that is, the pair (p_1, e^*) does not provide a terminating trajectory emanating from x. It suffices then to demonstrate that (p_1, e^*) does, in fact, induce a terminating trajectory, say ϕ_1.

Let $\phi^* \in T(x; p^*, e^*)$. Let $[t, t_f^*]$ be the time domain of ϕ^*, where t corresponds to the state x if time is a component of the state variable, say $x_n = t$. Let t_a be the smallest time contained in $[t, t_f^*]$ such that $\phi^*(t_a) \in A$. If no such t_a exists, then $\phi^* \in T(x; p_1, e^*)$ because $p_1(x) = p^*(x)$ for all $x \notin A$. Define $\phi_1 = \phi^*$. The nonexistence of t_a contradicts, therefore, the assumption that $(p_1, e^*) \notin \mathcal{T}(x)$.

Hence, suppose t_a exists. Let $x_a = \phi^*(t_a)$. Now let t_b be the maximum time such that $\phi(t_b) = x_a$. Note that if time is a component of the state variable, then $t_b = t_a$. We will consider this case first and afterward the case in which time does not enter explicitly into the dynamics; i.e., an autonomous process.

If $t_b = t_a$, define the absolutely continuous function ϕ_1 by

$$\phi_1(\tau) = \begin{cases} \phi^*(\tau) & \forall \tau \in [t, t_a] \\ \phi(\tau) & \forall \tau \in (t_a, t_f]. \end{cases} \tag{22}$$

Observe that ϕ_1 satisfies Eq. (6) for the strategy pair (p_1, e^*) where x is the initial state. Again we have shown that $(p_1, e^*) \in \mathcal{T}(x)$.

Suppose the process is autonomous. A trivial but important property of such processes is that if ϕ^* is a solution of Eq. (6), then so is ϕ_c, [9], where

$$\phi_c(\tau) = \phi^*(\tau + c) \qquad \tau \in [t - c, t_f^* - c] \tag{23}$$

and where c is any real number. Let

$$c = t_a - t_b \, . \tag{24}$$

Note that ϕ_c has the time domain $[t - t_a + t_b, t_f^* - t_a + t_b]$ and that $\phi_c(t_b) = \phi(t_a) = x_a$.

Define

$$\phi_1(\tau) = \begin{cases} \phi_c(\tau) & \forall\tau \in [t - c, t_b] \\ \phi(\tau) & \forall\tau \in (t_b, t_f] \, . \end{cases} \tag{25}$$

Thus, $\phi_1 \in T(x; p_1, e^*)$ and $(p_1, e^*) \in \mathcal{T}(x)$, concluding the proof of the lemma.

Interchanging the roles of p and e in this lemma yields an analogous result for constructing a playable strategy pair (p^*, e_1) from two other playable strategy pairs (p^*, e^*) and (p^*, e). This observation together with the above lemma is used in the proof of the following theorem.

Theorem. *A strategy pair* $(\mathrm{p}^*, \mathrm{e}^*)$ *that is playable over all of the playing space is a saddlepoint strategy of Type II iff it is a saddlepoint strategy of Type I.*

Proof. Let $(p^*, e^*) \in \mathcal{T}(X)$ be optimal according to the saddlepoint optimality criterion of Type I. From the definitions of $\mathcal{P}(e^*, X)$ and $\mathcal{E}(p^*, X)$ in Eqs. (10) and (11), it follows from inspection of Eqs. (13) to (16) that (p^*, e^*) is optimal of Type II.

Conversely, suppose (p^*, e^*) satisfies the saddlepoint optimality criterion of Type II. Suppose further that (p^*, e^*) is not optimal of Type I. This implies that there exist $x_0 \in X$, $p \in \mathcal{P}(e^*, x_0)$, $\phi^* \in T(x_0, p^*, e^*)$, and $\phi \in T(x_0; p, e^*)$ such that

$$V(x_0; p^*, e^*, \phi^*) > V(x_0; p, e^*, \phi); \tag{26}$$

or there exist, in place of p and ϕ, $e \in \mathcal{E}(p^*, x_0)$ and $\hat{\phi} \in T(x_0; p^*, e)$ such that

$$V(x_0; p^*, e, \hat{\phi}) > V(x_0; p^*, e^*, \phi^*) \, . \tag{27}$$

We show that (26) cannot hold. It can be shown similarly that (27) cannot hold. Using the definition of (17), we note that $\phi \in T(x_0; p_1, e^*)$ and that

$$V(x_0; p, e^*, \phi) = V(x_0; p_1, e^*, \phi) \, . \tag{28}$$

According to Lemma 1, $(p_1, e^*) \in \mathcal{T}(X)$. Thus, from (15) we have

$$V(x_0; p^*, e^*, \phi^*) \leqslant V(x_0; p_1, e^*, \phi) . \tag{29}$$

Inequality (29) together with (28) contradicts (26), concluding the proof.

DISCUSSION

Let S denote a set of strategies $\sigma : E^n \to W$ where W is compact in E^m. Consider the time interval $[0, 1]$ and let $TR([0, 1], E^n)$ denote a set of trajectories with domain $[0, 1]$ and range in E^n. Let $AC([0, 1], E^n]$ represent the set of all absolutely continuous functions from $[0, 1]$ into E^n. Because all solutions of ordinary differential equations are absolutely continuous and trajectories are such solutions, it follows necessarily that

$$TR([0, 1], E^n) \subset AC([0, 1], E^n).$$

Let ϕ belong to $TR([0, 1], E^n)$ and let $\phi([0, 1])$ denote the image of $[0, 1]$ under the mapping ϕ.

Definition. The set of strategies S is called *closed* with respect to $TR([0, 1], E^n)$ iff for every σ, $\sigma^* \in S$ and for every $\phi \in TR([0, 1], E^n)$, the function σ_1 also belongs to S, where

$$\sigma_1(x) = \begin{cases} \sigma(x) & \forall x \in \phi([0, 1]) \\ \sigma^*(x) & \forall x \in E^n - \phi([0, 1]). \end{cases} \tag{30}$$

If, in the above definition, $TR([0, 1], E^n)$ equals $AC([0, 1], E^n)$, then S is said to be closed. The property of a space S of strategies satisfying (30) is called a property of closure. It is shown in the proof of Lemma 1 that the set of Borel-measurable strategies is closed. It is this property of P and E being closed that led to the equivalence between the two types of saddlepoint optimality. If this closure property does not hold, it is conceivable that the two types of optimality could disagree on the optimality of a particular candidate in some game. True, if a playable strategy pair is optimal according to Type I, then it is also optimal according to Type II. The question is then whether it is possible for a strategy pair (p^*, e^*) to be optimal of Type II but not of Type I. This can only happen if (15) and (16) are met and if there are $x_0 \in X$, $\phi^* \in T(x_0; p^*, e^*)$, $p \in \mathcal{P}(e^*, x_0)$, and $\phi \in T(x_0; p, e^*)$ such that

$$V(x_0; p^*, e^*, \phi^*) > V(x_0; p, e^*, \phi), \tag{31}$$

or, in place of P and ϕ, there are $e \in \mathcal{E}(p^*, x_0)$ and $\hat{\phi} \in T(x_0; p^*, e)$ such that

$$V(x_0; p^*, e, \phi) > V(x_0; p^*, e^*, \phi^*). \tag{32}$$

A game in which (15), (16), and (32) hold is illustrated by the following simple example: Let

$$\dot{x}_1 = u, \qquad u \in U = \{-1, 1\} \tag{33}$$

$$\dot{x}_2 = v, \qquad v \in V = \{-1, 0\} \tag{34}$$

and let the integrand of the payoff, Eq. (12), be given by

$$f_0(x_1, x_2, u, v) = |u| - v\,. \tag{35}$$

Define

$$\theta_1 = \{x \in E^2 : x_2 = 0 \text{ and } x_1 \leqslant 0\},$$
$$\theta_2 = \{x \in E^2 : x_2 = 1 \text{ and } x_1 \geqslant 0\},$$
$$\theta_3 = \{x \in E^2 : x_1 = 0 \text{ and } 0 \leqslant x_2 \leqslant 1\},$$
$$X_1 = \{x \in E^2 : x_2 \geqslant 0 \text{ and } x_1 \leqslant 0\},$$
$$X_2 = \{x \in E^2 : x_2 \geqslant 1 \text{ and } x_1 \geqslant 0\}$$

where $x = (x_1, x_2)$. The playing space and target set are

$$X = X_1 \cup X_2$$
$$\theta = \theta_1 \cup \theta_2 \cup \theta_3.$$

In the game under discussion, we restrict the sets of admissible strategies, P and E, to the class of constants. Thus, $\hat{\mathscr{P}}$ and $\hat{\mathscr{E}}$ each contains only two strategies; namely,

$$\hat{\mathscr{P}} \text{ contains } p_i(x) = i, \qquad i = -1, 1$$
$$\hat{\mathscr{E}} \text{ contains } e_j(x) = j, \qquad j = -1, 0$$

for all $x \in X$.

The strategy pairs (p_{-1}, e_{-1}) and (p_1, e_{-1}) are the only pairs that are playable on the entire playing space X. We show subsequently that (p_1, e_{-1}) is optimal of Type II but not of Type I. In particular, we show that (27) holds.

Define

$$B_1 = \{x \in X | 0 \leqslant x_2 - 1 \leqslant x_1\},$$
$$B_2 = \{x \in X | x_1 < x_2 - 1\},$$
$$B_3 = \{x \in X | 0 \leqslant x_2 < 1, \ \ x_2 \leqslant x_1 + 1\}.$$

For $x_0 = (x_1(t_0), x_2(t_0))$, observe that

$$V(x_0; p_{-1}, e_{-1}) = \begin{cases} 2[x_2(t_0) - 1] & \forall x_0 \in B_1 \\ 2x_2(t_0) & \forall x_0 \in B_2 \cup B_3, \end{cases} \tag{36}$$

where the trajectory has been deleted as an argument of V because all solutions of Eqs. (33) and (34) are unique in this game.

Write

$$C_1 = \{x \in X | x_1 + x_2 \geqslant 1\},$$

$$C_2 = \{x \in X | x_1 + x_2 \leqslant 0\},$$

$$C_3 = \{x \in X | 0 \leqslant x_1 + x_2 \leqslant 1\}.$$

Note that

$$V(x_0; p_1, e_{-1}) = \begin{cases} 2[x_2(t_0) - 1] & \forall x_0 \in C_1 , \\ 2x_2(t_0) & \forall x_0 \in C_2 , \\ -2x_1(t_0) & \forall x_0 \in C_3 . \end{cases} \tag{37}$$

One can verify that

$$V(x_0; p_1, e_{-1}) \leqslant V(x_0; p_{-1}, e_{-1}) \qquad \forall x_0 \in X \tag{38}$$

and that, for all x_0 contained in the interior of $B_2 \cap C_1$,

$$V(x_0; p_1, e_{-1}) < V(x_0; p_{-1}, e_{-1}) . \tag{39}$$

The inequality (38) implies (15) is met for the candidate (p_1, e_{-1}) because (p_{-1}, e_{-1}) is the only other member of $\mathcal{P}(e_{-1}, X)$. The pair (p_{-1}, e_{-1}) is not optimal of Type II because (39) implies that (15) fails for this strategy.

The inequality (16) is trivially met for the candidate (p_1, e_{-1}) because e_{-1} is the unique member of $\mathcal{E}(p_1, X)$. Thus, the pair (p_1, e_{-1}) is optimal of Type II and as such it is unique.

To show that (p_1, e_{-1}) is not optimal of Type I, consider $\hat{x}_2 \in (0, 1)$. Select any $\hat{x}_1$ such that $\hat{x}_1 < -2\hat{x}_2$. Consider the effect of the strategy e_0 which is playable with p_1 at the initial state $\hat{x} = (\hat{x}_1, \hat{x}_2)$. Note that

$$V(\hat{x}; p_1, e_0) = -\hat{x}_1 .$$

Because $-\hat{x}_1 > 2\hat{x}_2$, we have

$$V(\hat{x}; p_1, e_0) > 2\hat{x}_2 . \tag{40}$$

Because $\hat{x} \in C_2$, it follows from (37) together with (40) that

$$V(\hat{x}; p_1, e_0) > V(\hat{x}; p_1, e_{-1}) . \tag{41}$$

This is the restatement of (32) for this game example. Therefore, the playable strategy pair (p_1, e_{-1}) is not optimal of Type I even though it is optimal of Type II.

Only constant strategies were admissible in this example, resulting in the nonequivalence of optimality Types I and II. For such strategies, the closure property does not hold. As shown by Theorem 1, the set of Borel strategies leads to an equivalence between the two optimality types, and these strategies satisfy the closure property. Nevertheless, the closure property, while being a sufficient condition for this equivalence, is by no means a necessary condition.

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